

KODAIRA TYPE VANISHING THEOREM FOR THE HIROKADO VARIETY

YUKIHIRO TAKAYAMA

ABSTRACT. The Hirokado variety is a Calabi-Yau threefold in characteristic 3 that is not liftable either to characteristic 0 or the ring W_2 of the second Witt vectors. Although Deligne-Illusie-Raynaud type Kodaira vanishing cannot be applied, we show that $H^1(X, L^{-1}) = 0$, for an ample line bundle such that L^3 has a non-trivial global section, holds for this variety. MSC Code: 14F17, 14J32, 14G17, 14M15.

1. INTRODUCTION

Although Calabi-Yau threefolds have the unobstructed deformation in characteristic 0 and every K3 surface in positive characteristic can be lifted to characteristic 0, the situation is quite different for Calabi-Yau threefolds in positive characteristic. Namely, some of Calabi-Yau threefolds in characteristic 2 or 3 cannot be lifted to characteristic 0 as shown in [10, 14, 11, 12, 1]. It is also known that there are un-liftable 3-dimensional Calabi-Yau algebraic spaces in many positive characteristics [3, 2].

On the other hand, it is well known that the liftability problem is closely related to Kodaira vanishing theorem. Namely, $W_2(K)$ -liftability, where $W_2(K)$ is the ring of the second Witt vectors, for varieties X over an algebraically closed field K of $\text{char}(K) = p > 0$ with $\dim X \leq p$ is a sufficient condition for Kodaira vanishing [6]. However it is not clear whether Kodaira vanishing holds without $W_2(K)$ -liftability.

As far as the author is aware, it is not known whether Kodaira vanishing holds on any of non-liftable Calabi-Yau threefolds in positive characteristic. In [15] the author studied possibility of constructing a Calabi-Yau threefold as a counterexample to Kodaira vanishing and showed that it is possible if there is a surface of general type with certain condition. However, we do not know if such a surface exists. Ekedahl [9] proved that both the Hirokado variety [10] and the Schröer variety [14] are not $W_2(K)$ -liftable. But this does not necessarily imply that Kodaira vanishing does not hold.

In this paper, we show that on the Hirokado variety Kodaira vanishing holds to some extent. Namely, we have $H^1(X, L^{-1}) = 0$ for any ample line bundle L such that L^3 has a non-trivial global section (Theorem 9). This means that there is an example that (a part of) Kodaira vanishing holds even if it is not $W_2(K)$ -liftable. The proof is a rather easy consequence of Ekedahl's interpretation of the Hirokado variety as a Deligne-Lusztig type variety associated to the Grassmannian $Gr(2, 4)$ together with the theory of pre-Tango structure [17, 18, 13, 16].

In the next section, we will briefly review the Ekedahl's reconstruction [9] of the Hirokado variety. We also summarize the theory of pre-Tango structure, which plays an essential role for construction of counterexamples to Kodaira vanishing. Then the main theorem will be proved in section 3.

2. PRELIMINARIES

In the following, K denotes an algebraically closed field of $\text{char}(K) = p > 0$.

2.1. Hirokado variety. Let $\mathbb{A}_K^3 := \text{Spec } K[x, y, z] \subset \mathbb{P}_K^3$ be an affine open subset. Consider the derivation δ on \mathbb{A}_K^3 :

$$(1) \quad \delta = (x^p - x) \frac{\partial}{\partial x} + (y^p - y) \frac{\partial}{\partial y} + (z^p - z) \frac{\partial}{\partial z}.$$

δ determines a vector field on \mathbb{P}_K^3 , which we will also denote by δ . δ is p -closed, i.e., $\delta^p = f\delta$ for some element f of the function field $K(X)$. Moreover, δ has $p^3 + p^2 + p + 1$ isolated singular points, namely $\delta = 0$ on $\mathbb{P}^3(\mathbb{F}_p) = \{[z_0 : z_1 : z_2 : z_3] \mid z_i^p = z_i, i = 0, 1, 2, 3\}$. Since these singular points are isolated they can be resolved by one point blow-ups. Now we obtain the following diagram.

$$(2) \quad \begin{array}{ccc} S & \xrightarrow{\psi} & X \\ \pi \downarrow & & \tilde{\pi} \downarrow \\ \mathbb{P}_K^3 & \xrightarrow{\psi'} & V \end{array}$$

where $\pi : S \rightarrow \mathbb{P}_K^3$ is the blow-up centered at $\mathbb{P}^3(\mathbb{F}_p)$, ψ' and ψ are the quotient maps by δ and $\pi^*\delta$ respectively. Namely, $\mathcal{O}_V = \{a \in \mathcal{O}_{\mathbb{P}_K^3} : \delta(a) = 0\}$ and $\mathcal{O}_X = \{a \in \mathcal{O}_S : \pi^*\delta(a) = 0\}$. $\tilde{\pi}$ is the naturally induced morphism. This variety X is Calabi-Yau only when $p = 3$, and we have the following non-liftability:

Theorem 1 (Corollary 2.3 [10] and Theorem A [9]). *If $p = 3$, Hirokado variety X is Calabi-Yau threefold that cannot be lifted to characteristic 0 or $W_2(K)$. Namely, there is no smooth projective morphism*

$$\psi : \mathfrak{X} \rightarrow \text{Spec } R,$$

where R is a discrete valuation ring of mixed characteristic or $R = W_2(K)$, whose special fiber is isomorphic to X . In particular, Deligne-Illusie-Raynaud type Kodaira vanishing does not hold on X .

2.2. Ekedahl's reconstruction of Hirokado variety.

2.2.1. Foliation and non-standard Gauss map. For a smooth variety X with $n = \dim X$ and its tangent sheaf \mathcal{T}_X , a subbundle $\mathcal{E} \subset \mathcal{T}_X$ of constant rank r that is closed under Lie brackets and p -th powers is called a *1-foliation*, on X . For simplicity, we will call it *foliation* in the following. For a purely inseparable finite flat morphism $f : X \rightarrow Y$ of degree p , the kernel $\mathcal{E} := \text{Ker } df$ of the differential of f is a foliation. In this case, f is called the *quotient map* by \mathcal{E} and we denote as $Y = X/\mathcal{E}$. Then we have $\mathcal{O}_Y \cong \{a \in \mathcal{O}_X \mid \delta(a) = 0 \text{ for all } \delta \in \mathcal{E}\}$. See, for example, [7, 8] for the detail about foliation.

Let \mathcal{G} be the Grassmannian bundle of r -dimensional subspaces of \mathcal{T}_X . By considering a coordinate neighborhood U ($\subset X$) that trivializes \mathcal{T}_X and thus \mathcal{G} , we can define a kind of Gauss map, which is the composition of the section

$$e : U \longrightarrow \mathcal{G} \quad \text{such that } U \ni x \longmapsto \mathcal{E}_x \in \mathcal{G}_x$$

and the projection $\mathcal{G}|_U \longrightarrow Gr(r, n) := \{V \subset K^n : \dim V = r\}$.

Now we consider the special case of $U = \mathbb{A}_K^n \subset X := \mathbb{P}_K^n$ together with a foliation \mathcal{E} of rank r on U that can be extended to X . By pulling back with the natural map $\varphi : \mathbb{A}^{n+1} - \{0\} \rightarrow \mathbb{P}^n$, we obtain the foliation $\varphi^*\mathcal{E}$ on $\varphi^{-1}(U)$, which induces the Gauss map in our sense

$$g' : \varphi^{-1}(U) \longrightarrow Gr(r+1, n+1).$$

Since this map is invariant under the \mathbf{G}_m -action, we obtain the new Gauss map

$$g : U \longrightarrow Gr(r+1, n+1)$$

which can be extended to $X = \mathbb{P}_K^n$. Moreover, we have

Proposition 2 (cf. Proposition 2.2 [9]). *The Gauss map $g : \mathbb{P}_K^n \rightarrow Gr(r+1, n+1)$ factors through the quotient map $\mathbb{P}_K^n \rightarrow \mathbb{P}_K^n/\mathcal{E}$ by \mathcal{E} . In particular, there exists the induced Gauss map*

$$g_{\mathcal{E}} : \mathbb{P}_K^n/\mathcal{E} \longrightarrow Gr(r+1, n+1)$$

such that g is the composition of the quotient map and $g_{\mathcal{E}}$.

2.2.2. Hirokado foliation on \mathbb{P}_K^n . Now we consider the case of $n = 3$ and $r = 1$. Let \mathcal{E} be the foliation on \mathbb{A}_K^3 and on \mathbb{P}_K^3 generated by the derivation δ as in (1). Then we have

$$\varphi^*\mathcal{E} = \left\langle \sum_{i=1}^4 X_i^p \frac{\partial}{\partial X_i}, \sum_{i=1}^4 X_i \frac{\partial}{\partial X_i} \right\rangle$$

where the natural map $\varphi : \mathbb{A}^4 - \{0\} \rightarrow \mathbb{P}^3$ is defined by $\varphi((X_1, X_2, X_3, X_4)) = [X_1 : X_2 : X_3 : X_4]$. Then by Proposition 2 we obtain the Gauss map $g' : \mathbb{P}_K^3 \setminus \mathbb{P}_K^3(\mathbb{F}_p) \rightarrow Gr(2, 4)$ and also $g_{\mathcal{E}'} : V \rightarrow Gr(2, 4)$ such that $g' = g_{\mathcal{E}'} \circ \psi'$. Moreover, these Gauss maps can be extended to the blown-up varieties and we have the Gauss maps

$$\hat{g} : S \longrightarrow Gr(2, 4) \quad \text{and} \quad g : X \longrightarrow Gr(2, 4)$$

such that $\hat{g} = g \circ \psi$. (Lemma 2.3 [9]).

2.2.3. the Hirokado variety as a Deligne-Lusztig type variety. Now we consider a subvariety \mathcal{F} of the Grassmannian $Gr(2, 4)$ defined as

$$\mathcal{F} = \{W \subset K^4 : \dim W = 2, W \cap F^*W \neq \emptyset\}$$

and the flag variety

$$\tilde{\mathcal{F}} = \{L \subset W \subset K^4 : \dim L = 1, \dim W = 2, L \subset F^*W\}$$

where $F : K^4 \longrightarrow K^4$ is the Frobenius morphism. We have the forgetful morphism

$$\Psi : \tilde{\mathcal{F}} \longrightarrow \mathcal{F} \quad \text{such that } (L \subset W \subset K^4) \longmapsto (W \subset K^4).$$

Then we have

Proposition 3 (Proposition 2.4 [9]). *\mathcal{F} contains the image of V by the Gauss map $g_{\mathcal{E}}$, $\tilde{\mathcal{F}} \cong X$ and Ψ is a desingularization of \mathcal{F} , whose exceptional set has codimension 2.*

It is well known that $Gr(2, 4)$ is the hypersurface in \mathbb{P}_K^5 defined by the Plücker relation

$$q(X) := X_{12}X_{34} - X_{12}X_{24} + X_{14}X_{23}$$

where X_{ij} ($1 \leq i < j \leq 4$) are the homogeneous coordinates of \mathbb{P}_K^5 . On the other hand, we can regard $Gr(2, 4)$ as the set of projective lines in \mathbb{P}_K^3 . Thus every element $W \in \mathcal{F}$ can be regarded as a projective line in \mathbb{P}_K^3 that intersects with the projective line represented by F^*W . This relation is described by the bilinearization of the Plücker relation

$$b(X, X^p) = X_{12}X_{34}^p - X_{12}X_{24}^p + X_{14}X_{23}^p + X_{34}X_{12}^p - X_{24}X_{13}^p + X_{23}X_{14}^p,$$

which is a $(p+1)$ -form (see, for example, Proposition 12.1.1 [4]). Thus \mathcal{F} is a complete intersection of type $(2, p+1)$ in \mathbb{P}_K^5 .

Consequently, we have the following.

Theorem 4 (T. Ekedahl [9]). *The Hirokado variety X is obtained from a complete intersection of type $(2, 4)$ in \mathbb{P}_K^5 by blowing up centered at the 40 points in $\mathbb{P}_{\mathbb{F}_3}^5$.*

Now we consider the Hodge cohomologies of the Hirokado variety.

Proposition 5 (cf. Theorem 3.6(i) and Proposition 3.1(iii) [9]). $\Psi_*\Omega_{\tilde{\mathcal{F}}}^1 = \Omega_{\mathcal{F}}^1$.

Proof. Let $j : U \hookrightarrow \mathcal{F}$ be the inclusion of the non-singular locus. Since $\Psi : \tilde{\mathcal{F}} \rightarrow \mathcal{F}$ is a small resolution and its exceptional set has codimension 2, we have $\Psi_*\Omega_{\tilde{\mathcal{F}}}^1 = j_*\Omega_U^1$. Moreover, since $\mathcal{F} - U$ has codimension ≥ 2 , we can define $\Omega_{\mathcal{F}}^1$ as the natural extension of $j_*\Omega_U^1$. Thus we have $\Psi_*\Omega_{\tilde{\mathcal{F}}}^1 = \Omega_{\mathcal{F}}^1$. \square

The cohomologies of smooth complete intersections have been computed in Proposition 1.3 of [5]. This result has been extended to the case of singular complete intersection by Ekedahl (Proposition 1.2 [9]), from which we obtain the following.

Proposition 6 (Corollary 1.3 [9]). *Let Y be an n -dimensional complete intersection in \mathbb{P}_K^r with only isolated singularities. For the Hodge numbers $h_Y^{ij} := \dim H^j(Y, \Omega_Y^i)$ we have that $h^{ij} = \delta_{ij}$ when $i + j < n$.*

Corollary 7. $H^0(\mathcal{F}, \Omega_{\mathcal{F}}^1) = 0$.

2.3. Tango structure and Kodaira vanishing. Let X be a smooth projective variety. Then an ample divisor D , or an ample line bundle $L = \mathcal{O}_X(D)$, is called a pre-Tango structure if there exists an element $\eta \in K(X) \setminus K(X)^p$ such that the Kähler differential is $d\eta \in \Omega_X^1(-pD)$, which will be simply denoted as $(dy) \geq pD$.

If there exists a pre-Tango structure $L = \mathcal{O}_X(D)$, we have $H^1(X, L^{-1}) \neq 0$. In fact, consider the absolute Frobenius map $F : \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X(-pD)$ and set $B_X(-D) := \text{Coker } F$. Then we have the exact sequence

$$0 \longrightarrow H^0(X, B_X(-D)) \longrightarrow H^1(X, \mathcal{O}_X(-D)) \xrightarrow{F} H^1(X, \mathcal{O}_X(-pD))$$

where we have $H^0(X, B_X(-D)) = \{df \in K(X) \mid (df) \geq pD\}$ and this is non-trivial since D is pre-Tango. Notice that multiplication by a non-trivial element from $H^0(X, B_X(-D))$ gives an embedding $\mathcal{O}_X(pD) \hookrightarrow \Omega_X^1$. See, for example, [16, 17, 18, 13] for more detail information on the pre-Tango structure.

The inclusion $H^0(X, B_X(-D)) \subset H^1(X, \mathcal{O}_X(-D))$ may be strict, which means that, for an ample line bundle L , $H^1(X, L^{-1}) \neq 0$ does not always mean that L is a pre-Tango structure. However, we have

Proposition 8. *If $H^1(X, L^{-1}) \neq 0$ for an ample line bundle L , then L^n is a pre-Tango structure for some integer $n \geq 1$.*

Proof. Since X is normal, Enriques-Severi-Zariski's theorem shows that iterated Frobenius maps

$$F^e : H^1(X, L^{-1}) \longrightarrow H^1(X, L^{-p^e}) \quad (e \gg 0)$$

are trivial. Thus we have $H^0(X, B_X(-nD)) = H^1(X, L^{-n}) (\neq 0)$ for a sufficiently large $n \in \mathbb{N}$, or precisely for $n = p^e$ such that $H^1(X, L^{-p^e}) \neq 0$ but $H^1(X, L^{-p^{e+1}}) = 0$. \square

3. MAIN THEOREM

Now we can prove the main result.

Theorem 9. *Let X be the Hirokado variety and L be an ample line bundle with $H^0(X, L^3) \neq 0$. Then we have $H^1(X, L^{-1}) = 0$.*

Proof. Assume that $H^1(X, L^{-1}) \neq 0$ for some ample line bundle L . Then by Proposition 8 there exists an integer $n \geq 1$ such that $\mathcal{L} = L^n$ is a pre-Tango structure. Thus we have $\mathcal{L}^p \subset \Omega_X^1$, $p = 3$

On the other hand, by Proposition 3 and Theorem 4, we have $X \cong \tilde{\mathcal{F}}$ together with a desingularization $\Psi : \tilde{\mathcal{F}} \rightarrow \mathcal{F}$ of a $(2, 4)$ -type complete intersection \mathcal{F} in \mathbb{P}_K^5 . Then we compute

$$H^0(\tilde{\mathcal{F}}, \Omega_{\tilde{\mathcal{F}}}^1) = H^0(\mathcal{F}, \Psi_* \Omega_{\tilde{\mathcal{F}}}^1) = H^0(\mathcal{F}, \Omega_{\mathcal{F}}^1) = 0$$

by Proposition 5 and Corollary 7. Since $H^0(\tilde{\mathcal{F}}, \mathcal{L}^p) \subset H^0(\tilde{\mathcal{F}}, \Omega_{\tilde{\mathcal{F}}}^1)$, we then have $H^0(X, L^{np}) = H^0(\tilde{\mathcal{F}}, \mathcal{L}^p) = 0$, which contradicts the assumption $H^0(X, L^p) \neq 0$, $p = 3$. \square

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YUKIHIRO TAKAYAMA, DEPARTMENT OF MATHEMATICAL SCIENCES, RITSUMEIKAN UNIVERSITY, 1-1-1 NOJIHIGASHI, KUSATSU, SHIGA 525-8577, JAPAN
E-mail address: takayama@se.ritsumei.ac.jp